The Iteration Number of the Weisfeiler-Leman Algorithm, Grohe et al. (2023)

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Linear Algebra Methods in Combinatorics

cf. Babai and Frankl (1988)

Definition 1

Let L be a set of nonnegative integers. The family $\mathcal{F}$ is L-intersecting, if $|E \cap F| \in L$ for every pair E, F of distinct members of $\mathcal{F}$.

Problem 2 (Restricted Intersection Problem)

What is the maximum number of members in a k-uniform L-intersecting family of subsets of a set of n elements?

Couple theorems

Theorem 3 (Ray-Chaudhuri – Wilson Theorem)

Let L be a set of s integers and $\mathcal{F}$ an L-intersecting k-uniform family of subsets of a set of n elements, where $s \leq k$. Then:

$$|\mathcal{F}| \leq \binom{n}{s}$$

Theorem 4 (Babai and Frankl (1988))

For every $k \geq s \geq 1$ and $n \geq 2k^2$, there exists a k-uniform family $\mathcal{F}$ of size:

$$|\mathcal{F}| > (n/2k)^s$$

on n points such that $|E \cap F| \leq s - 1$ for any two distinct sets $E, F \in \mathcal{F}$.

Proof of this second theorem

Proof.

Let p be a prime st $n/(2k) < p \leq n/k$, and we have $k \leq p$. Fix a k-subset A of $\mathbb{F}_p$. Let X be an n-set containing $A \times \mathbb{F}_p$. For a function $f : A \to \mathbb{F}_p$, the graph

$$G(f) = \{(\xi, f(\xi)) : \xi \in A\}$$

is a k-subset of X.

Let $\mathcal{F}$ consist of the graphs of the polynomials of degree $\leq s-1$ over $\mathbb{F}_p$, restricted to A. Any two different polynomials of degree$\leq s-1$, must have their graphs with at most $s-1$ points in common.

The number of such polynomials is $p^s > (n/2k)^s$. $\square$

Invariants, colouring and the WL algorithm

Definition 5 (Colouring algorithm (or 1-WL))

For every graph G, we define a sequence of vertex colourings $\chi_1^{(t)}(G)$ as follows:

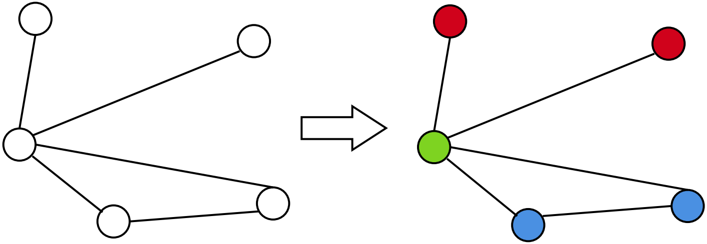
- For every $v \in V(G)$, let $\chi_1^{(0)}(G, v) := col(G, v)$.

$$\chi_1^{(t+1)}(G, v) := (\chi_1^{(t)}(G, v), \{\{\chi_1^{(t)}(G, w) | w \in N(v)\}\})$$

- The stable colouring is denoted $\chi_1^{(\infty)}(G)$.
- $\chi_1^{(t)}$ is vertex invariant.

In the same way we can define the k-dimensional WL algorithm $\chi_k : V^k \to C$.

An example



$$\chi_1^{(t+1)}(G, v) := (\chi_1^{(t)}(G, v), \{\{\chi_1^{(t)}(G, w) | w \in N(v)\}\})$$

WL algorithm

Let V be a finite set and let $\chi : V^k \to C$ be a coloring.

- $\mathit{step}_k(\chi)(v) := (\chi(v), M_\chi(v))$, where

  $$M_\chi(v) = \{\{(\chi(v[w/1]), \ldots, \chi(v[w/k]))|w \in V\}\}$$

- $\mathit{atp}_{\mathfrak{A}}(v)$ is the isomorphism type of the ordered substructure of $\mathfrak{A}$ induced by $\{v_1, \ldots, v_k\}$.

Definition 6 (k-WL algorithm)

- $\chi_k^{(0)}[\mathfrak{A}](v) := \mathit{atp}_{\mathfrak{A}}(v)$
- $\chi_k^{(r+1)}[\mathfrak{A}] := \mathit{step}_k(\chi_k^{(r)}[\mathfrak{A}])$, stabilizes at $\chi_k^{(\infty)}[\mathfrak{A}]$.

An upper bound on the number of iterations

The coloring χ_1 refines χ_2, denoted $\chi_1 \preceq \chi_2$, if χ_1 is a more intricate coloring than χ_2.

Theorem 7

Let V be a finite set of size $n := |V|$, let $\chi_0 \dots, \chi_\ell : V^k \to C$ be a sequence of colorings such that:

- *χ_t is shufflable and compatible with equality for all $t \in [0, \ell]$,*
- *$step_k(\chi_{t-1}) \succeq \chi_t$ for all $t \in [\ell]$, and*
- *$\chi_{t-1} \succ \chi_t$ for all $t \in [\ell]$.*

Then $\ell \leq 2n^{k-1}(\lceil k \log n \rceil + 1) = O(kn^{k-1} \log n)$.

A long sequence of stable coloring

Theorem 8

Suppose $n \geq 2k^2$ and let V be a set of size $|V| = 2n$, then there is a sequence of colorings $\chi_0 \ldots, \chi_\ell : V^k \to C$ of length $\ell \geq (\frac{n}{2k})^{k-1}$ such that:

- *χ_t is shufflable and compatible with equality for all $t \in [0, \ell]$,*
- *χ_t is stable for all $t \in [0, \ell]$, and*
- *$\chi_{t-1} \succ \chi_t$ for all $t \in [\ell]$.*

How to get such colorings?

From $\mathcal{F}$ on n points, we induce a coloring $\chi_{\mathcal{F}}$ on $V = U \times \{0,1\}$, as follows:

$((u_1, a_1), \ldots, (u_k, a_k))$, $((u'_1 a'_1), \ldots, (u'_k, a'_k))$ are the same color iff:

(A) $u_i = u'_i$ for all $i \in [k]$,

(B) $(u_i, a_i) = (u_j, a_j) \Leftrightarrow (u'_i, a'_i) = (u'_j, a'_j)$ for all $i, j \in [k]$, and

(C) if $\{u_1, \ldots, u_k\} \in \mathcal{F}$, then $\sum_i a_i \equiv \sum_i a'_i \bmod 2$.

Proof of theorem 8

Recall Theorem 4:
For every $n \geq 2k^2$ there exists a k-uniform set family $\mathcal{F}$ over a universe U of n points such that:

- $|E_1 \cap E_2| \leq k - 2$ for all distinct $E_1, E_2 \in \mathcal{F}$, and
- $|\mathcal{F}| \geq (\frac{n}{2k})^{k-1}$.

We set $\mathcal{F}_t := \{E_1, \ldots, E_t\}$ and $\chi_t = \chi_{\mathcal{F}_t}$.
We get stability (hard part), refinement, and compatible with equality, shufflable properties of this sequence of colouring.

References

Babai, L. and Frankl, P. (1988). Linear algebra methods in combinatorics i.

Grohe, M., Lichter, M., and Neuen, D. (2023). The iteration number of the weisfeiler-leman algorithm.