Probabilities on finite models and graphs 0-1 laws

Gregoire Fournier

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1 Probabilities on Graphs [3]

We consider undirected graphs. By G_{R_n} we denote the set of all graphs with the universe $\{0, \dots, n-1\}$. The number of undirected graphs on $\{0, \dots, n-1\}$ is $|G_{R_n}| = 2^{\binom{n}{2}}$. Let $\mathcal{P}$ be a property of graphs. We define:

$$\mu_n(\mathcal{P}) = \frac{|\{G \in G_{R_n} | G \text{ has } \mathcal{P}\}|}{|G_{R_n}|}$$

The asymptotic probability of $\mathcal{P}$, if it exists, is denoted by:

$$\mu(\mathcal{P}) = \lim_{n\to\infty} \mu_n(\mathcal{P})$$

We give a couple of examples:

- $\mathcal{P}$:= "there are no isolated nodes":

$$\mu_n(\mathcal{P}) \leq \frac{n\, 2^{\binom{n-1}{2}}}{2^{\binom{n}{2}}} \to 0$$

- $\mathcal{P}$:= "the graph is connected":

$$\mu_n(\mathcal{P}) \leq \sum_{k=1}^{n-1} \frac{\binom{n}{k}\, 2^{\binom{k}{2}}\, 2^{\binom{n-k}{2}}}{2^{\binom{n}{2}}} \to 0$$

- $\mathcal{P}$:= "the graph has an even number of nodes":

$$\mu_n(\mathcal{P}) = \begin{cases} 0 \text{ if } n \text{ odd} \\ 1 \text{ if } n \text{ even} \end{cases}$$

  Therefore $\mu(\mathcal{P})$ does not exist.

2 0-1 laws [1]

We say that a logic has the zero-one law if for every property $\mathcal{P}$ definable in the logic, either $\mu(\mathcal{P}) = 0$, or $\mu(\mathcal{P}) = 1$.

Theorem 1. First order logic has the has the zero-one law.

2.1 Extension axioms

The proof relies on the extension axioms, defined as:

$$EA_{k,m} : \forall x_1, \ldots, x_k \left(\bigwedge x_i \neq x_j \right) \Rightarrow \exists y \begin{pmatrix} \bigwedge y \neq x_i \\ \wedge \bigwedge_{i \leq m} E(y, x_i) \\ \wedge \bigwedge_{i > m} \neg E(y, x_i) \end{pmatrix} \tag{1}$$

Claim 2. $\mu(EA_k) = 1$ for all k.

Proof. We show that $\mu(\neg EA_k) = 0$ for all k. Calling X and Y the two sets of size k, we count how we can choose X and Y for a graph of size n:

- We can choose X in $\binom{n}{k}$ different ways and Y in $\binom{n-k}{k}$ different ways.
- There are $2^{\binom{2k}{2}}$ ways to put edges on $X \cup Y$ and $2^{\binom{n-2k}{2}}$ ways to put edges on $\overline{X \cup Y}$.
- For each element $z \in \overline{X \cup Y}$, we can put edges between z and the $2k$ elements of $X \cup Y$ in every possible way except if z is connected to every member of X and not to any member of Y, which is one way. So in total we get $(2^{2k} - 1)^{n-2k}$ different ways to connect the partition.

So

$$\mu(\neg EA_k) \leq \frac{\binom{n-k}{k} 2^{\binom{2k}{2}} 2^{\binom{n-2k}{2}} (2^{2k} - 1)^{n-2k}}{2^{\binom{n}{2}}} \to 0$$

□

We get several corollaries:

- $\mu(\neg EA_{k,m}) = 1$ for $m \leq k$.
- Almost all graphs satisfy EA_k.
- Since $EA_k := EA_{2k,k}$ implies $EA_{k,m}$ for $m \leq k$, almost all graphs satisfy $EA_{k,m}$.

2.2 Pebbling game

Definition 3 (EF k-pebbling game). The game $EF_k(A, B)$ is played on two relational structures A and B. There are two players, Spoiler and Duplicator, and k pairs of pebbles (a_i, b_i) for $i \in [k]$. We denote the positions by $\vec{a} = (a_1, \ldots, a_k)$, $\vec{b} = (b_1, \ldots, b_k)$. Each move goes as follows:

- Spoiler chooses a structure, A or B, and a number $1 \leq i \leq k$.
- Spoiler places the pebble a_i on some element of A.
- Duplicator responds by placing b_i.

Duplicator has a winning strategy if he can ensure that after every round $((\vec{a}, \vec{c}^{\mathcal{A}}), (\vec{b}, \vec{c}^{\mathcal{B}}))$ is a partial isomorphism between A and B, where $\vec{c}$ denotes the constants of the language.

Theorem 4. [2] The following are equivalent:

- A and B satisfy the same sentences of FO using at most k variables.
- Duplicator has a winning strategy in the $EF_k(A, B)$ pebbling game, and we write $A \equiv_k B$.

2.3 0-1 Law on first order logic

Theorem 5. Let G_1, G_2 be finite graphs such that G_1, G_2 verify EA_k. Then $G_1 \equiv_k G_2$.

Proof. The language of graphs is only composed of the E binary relation, without any constant. Therefore a partial isomorphism only looks at the edge relations between the nodes picked.
Since $EA_k := EA_{2k,k}$ implies $EA_{k,m}$ for $m \leq k$, Duplicator can continue indefinitely to match Spoiler's choice, so $G_1 \equiv_k G_2$. $\square$

Now we prove that First Order logic has the zero-one law property:

Proof. Let ϕ be a FO formula with k variables.

- Suppose G is a graph with at least $2k$ elements satisfying both ϕ and EA_k. For any G' that satisfies EA_k and has at least $2k$ elements, Theorem 5 imply that $G \equiv_k G'$ and then Theorem 4 shows that G' satisfies ϕ. Therefore, $\mu(\phi) \geq \mu(EA_k) = 1$.
- Conversely, assume that no graph satisfying EA_k of size at least $2k$ also verifies ϕ. Then $\mu(\phi) \geq \mu(\neg EA_k) = 0$.

$\square$

3 Additional results

3.1 Graphs up to isomorphism

Let ν_n be the probability measure induced on the set of graphs of size n where each isomorphism class is represented by only one of its member.

Theorem 6. For ϕ is a first-order sentence, then $\nu_n(\phi)$ converges as $n \to \infty$, and $\lim_{n\to\infty} \nu_n(\phi) = \lim_{n\to\infty} \mu_n(\phi)$.

3.2 Random graph

For an edge probability function $p(n)$, we derived an induced sequence of probability spaces based on $G(n, p(n))$.
The case $p = 1/2$ corresponds to part 2.

Definition 7. We say that an edge relation $p(n)$ satisfies the zero-one law if, for every first order sentence ϕ:

$$\lim_{n\to\infty} P[G(n, p(n)) \vDash \phi] \in \{0, 1\}$$

Theorem 8 ([5]). If $0 < \alpha < 1$ and α is irrational then $p(n) = n^{-\alpha}$ satisfies the zero-one law.

3.3 Other random graph models

Theorem 9 ([4]). Let G_n be a random graph following a Barabasi Albert model of parameter m. Then G_n follows a FO^{m-2} convergence law.

References

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