MATH 539 lecture notes: Ultrafilters and Tychonoff 's compactness theorem [3]

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MATH 539: functional analysis

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Nets and filters

Recall what nets are:

Definition (Net)

A subset $\{x_\alpha\}_{\alpha \in A} \subset X$ is called a net if the index set A is partially ordered and (upward) directed. A subnet is a net $\{y_\beta\}_{\beta \in B}$ with a map $n : B \to A$ st. $y_\beta = x_{n(\beta)}$, n is monotone, and for every $\alpha \in A$ there is $\beta \in B$ with $n(\beta) \geq \alpha$.

A net $\{x_\alpha\}_{\alpha \in A}$ is convergent to $x \in X$, $x = lim_{\alpha \in A}\ x_\alpha = lim\ x_\alpha$, if for very open neighbourhood U of x, there is $\alpha_0 \in A$ st. $x_\alpha \in U$ for all $\alpha \geq \alpha_0$.

Lemma

$K \subset X$ is compact iff every net on K contains a convergent subnet to a point in K.

Filters

Definition (filter)

Suppose I is a non-empty set, $\mathcal{F} \subseteq \mathcal{P}(I)$ is called a filter on I if :

(i) $I \in \mathcal{F}$, $\emptyset \notin \mathcal{F}$.

(ii) If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$.

(iii) If $A \subseteq B$ and $A \in \mathcal{F}$ then $B \in \mathcal{F}$.

Some well known examples are:

- $\{X : \mathbb{R} \setminus X$ has Lebesgue measure zero$\}$
- If $|I| \geq \kappa$, $\{X : |I \setminus X| < \kappa\}$.
- Frechet filter: $\{X : |I \setminus X| < \omega\}$ for I an infinite set.
- Example of principal (filter has a $\subseteq$ minimum): $\{A \subseteq I : x \in A\}$ for some $x \in I$.

Ultrafilters

Note that the sets of filter can be ordered by the inclusion property.
Such a poset satisfies the conditions of Zorn's lemma as we can define for each chain a maximal element.

Definition (ultrafilter)

An ultrafilter is a maximal filter (through the poset of filters).

Lemma

(iv) $\mathcal{U}$ *is an ultrafilter iff* $\forall A \subseteq I$, *either* $A \in \mathcal{U}$ *or* $I \setminus A \in \mathcal{U}$.

Which is equivalent to:

(iv') If $\bigcup_{i=1}^{n} A_i \in \mathcal{U}$, then some $A_i \in \mathcal{U}$.

Ultrafilters

Lemma

$\mathcal{U}$ is an ultrafilter iff $\forall A \subseteq I$, either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$.

Proof.

$\Leftarrow$: Suppose $\mathcal{U}$ is not a maximal, i.e there exists $\mathcal{F} \supsetneq \mathcal{U}$. Consider $A \in \mathcal{F} \setminus \mathcal{U}$, by assumption we must have $I \setminus A \in \mathcal{U}$, but since $I \in \mathcal{U}$ and $\mathcal{U}$ is closed under $\cap$, $A \in \mathcal{U}$, which is a contradiction.

$\Rightarrow$: Given $\mathcal{U}$ an ultrafilter and $A \notin \mathcal{U}$ define:

$$\mathcal{F}_A = \{X \in I : (Y \setminus A) \subseteq X \text{ for some } Y \in \mathcal{U}\}$$

It is not too hard to check that $\mathcal{F}_A$ is a filter on I that contains $I \setminus A$ and $\mathcal{U}$. Therefore $I \setminus A \in \mathcal{F}_A = \mathcal{U}$.

□

Compactness and ultrafilters

The compactness of a topological space can be restated in terms of convergence of ultrafilters:

- For (X, τ) a topological space and $\mathcal{F}$ a filter on it, we say that $lim\ \mathcal{F} = x$ if every neighbourhood of x has a non-empty intersection with all of the elements of $\mathcal{F}$.
- Furthermore, if $\mathcal{F}$ is an ultrafilter, every set that intersects every element of $\mathcal{F}$ must lie in $\mathcal{F}$. Then, $lim\ \mathcal{F} = x$ iff every open neighbourhood of x is contained in $\mathcal{F}$.

If every two distinct points in X can be separated by disjoint open neighbourhoods, the space is called Hausdorff, and the limit is unique.

Compactness and ultrafilters

Lemma

X is compact if and only if every ultrafilter on X converges to a point in X.

Proof.

$\Rightarrow$: Suppose X is compact, and let $\mathcal{U}$ be an ultrafilter on X. If $\mathcal{U}$ does not converge to any point in X, then for any point $x \in X$, x contained in $U_x \in \tau$, with $U_x \notin \mathcal{U}$. Thus, $\{U_x\}_{x \in X}$ forms an open cover of X, which must contain a finite subcover, $U_1, \ldots, U_n$. But $\bigcup_j^n U_j = X \subset \mathcal{U}$, so one U_i must be in $\mathcal{U}$ by (iv'), a contradiction.

□

Compactness and ultrafilters

Lemma

X is compact if and only if every ultrafilter on X converges to a point in X.

Proof.

$\Leftarrow$: Let $\mathcal{C} = \{U_i : i \in I\}$ be an open cover of X. Suppose that it contains no finite subcover, i.e any finite intersection $(X \setminus U_1) \cap \ldots \cap (X \setminus U_n)$ is non-empty.
Therefore $\mathcal{F} = \{F \subset X : X \setminus U \subset F$, for some $U \in \mathcal{C}\}$ is a filter. Consider $\mathcal{U}$ an ultrafilter containing $\mathcal{F}$, by assumption, let $x = lim\, \mathcal{U}$. Since $\mathcal{C}$ is a cover, there is $U_i \in \mathcal{C}$ with $x \in U_i$. Finally, $X \setminus U_i \in \mathcal{U}$ and since $x = lim\, \mathcal{U}$, $U_i \in \mathcal{U}$, a contradiction.

□

Product topology

Let $\{(X_\gamma, \tau_\gamma)\}_{\gamma\in\Gamma}$ be a collection of topological spaces. The Cartesian product $X = \prod_{\gamma\in\Gamma} X_\gamma$ is the set of functions $x : \Gamma \to \bigcup_{\gamma\in\Gamma} X_\gamma$ st. $x(\gamma) \in X\gamma$. We write $x(\gamma) = x_\gamma$ and $x = \{x_\gamma\}_{\gamma\in\Gamma}$. Let $\pi_\gamma : X \to X_\gamma$ be the usual projection map. We define the product topology on the Cartesian product X to be the topology generated by the sets $\pi_\gamma^{-1}(U_\gamma)$, where $U_\gamma \in \tau_\gamma$. It is interesting to note that:

$$\{\bigcap_{i\in F} \pi_{\gamma_i}^{-1}(U_i) : \text{ where } F \text{ is finite and the } U_i \text{ are open in } X_{\gamma_i}\}$$

is a base for the product topology.

Product topology

Some remarks:

- The product topology is the minimal topology on X in which all the projection maps are continuous.
- In the same way, the weak* topology is the minimal topology on X^* in which the $J(x)(x^*) = x^*(x)$ are continuous.
- The weak topology is the minimal topology on X in which elements of X^* are continuous.

Tychonoff's compactness theorem

Theorem (Tychonoff's compactness theorem)
If all $X_\gamma, \gamma \in \Gamma$, are compact, then the Cartesian product $X = \prod_{\gamma \in \Gamma} X_\gamma$ is compact in the product topology.

Tychonoff's compactness theorem

Proof.

Let $\mathcal{U}$ be an ultrafilter on X. For any $\gamma \in \Gamma$, consider $\mathcal{U}_\gamma = \pi_\gamma(\mathcal{U})$. It is easy to check that $\mathcal{U}_\gamma$ is an ultrafilter on X_γ. Since X_γ is compact, there exists a limit $x_\gamma = lim\,\mathcal{U}_\gamma$.
Let us show that $x = \{x_\gamma\}_{\gamma \in \Gamma} = lim\,\mathcal{U}$.
Take U an open neighbourhood of x in the product topology. Then along with x, U contains a finite intersection of basis sets $\bigcap_{i=1}^{n} \pi_{\gamma_i}^{-1}(U_i)$. We have $x_{\gamma_i} \in U_i$, and hence $U_i \in \mathcal{U}_{\gamma_i}$, which means there exist $A_i \in \mathcal{U}$ such that $\pi_{\gamma_i}(A_i) = U_i$. Then $A_i \subset \pi_{\gamma_i}^{-1}(U_i)$, which implies that $\pi_{\gamma_i}^{-1}(U_i) \in \mathcal{U}$, for each i, and finally $\bigcap_{i=1}^{n} \pi_{\gamma_i}^{-1}(U_i) \in \mathcal{U}$.
Since U contains that intersection, U must be in the ultrafilter $\mathcal{U}$. □

Ultrafilters in logic (set theory and model theory)[1][4][2]

- Short proof of the Compactness Theorem using Los's Theorem on ultraproducts.
- Ideals, dual of filter, close downwards and under finite union.
- a club: $C \subseteq \kappa$ is a club if it is closed and unbounded in κ, club filter is $Ce(\kappa) = \{A \subset \kappa |$ club $C, C \subset A\}$ where κ is a regular uncountable cardinal.
- Cohen Forcing uses $\mathbb{P}$-generic filters over a model to show CON(ZFC + $\neg$ CH).

References

[1] D. Marker. Math 502 model theory lecture notes, 2015.

[2] J. Nagloo. Math 506 model theory lecture notes, 2022.

[3] R. Shvydkoy. Math 539 functional analysis lecture notes, 2022.

[4] D. Sinapova. Math 504 set theory lecture notes, 2022.